

CONSTRUCTING MASAS WITH PRESCRIBED PROPERTIES

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ABSTRACT. We consider an iterative procedure for constructing maximal abelian $*$ -subalgebras (MASAs) satisfying prescribed properties in II_1 factors. This method pairs well with the intertwining by bimodules technique and with properties of the MASA and of the ambient factor that can be described locally. We obtain such a local characterization for II_1 factors M that have an s -MASA, $A \subset M$ (i.e., for which $A \vee JAJ$ is maximal abelian in $\mathcal{B}(L^2M)$), and use this strategy to prove that any factor in this class has uncountably many non-intertwinable singular (respectively semiregular) s -MASAs.

0. INTRODUCTION

Given a separable II_1 factor M , one can construct a *maximal abelian $*$ -subalgebra* (abbreviated hereafter as *MASA*) A in M as an inductive limit of finite partitions. This iterative procedure pairs well with properties of MASAs that can be characterized locally, allowing the construction of A in a manner that makes “more and more” of the desired properties be satisfied.

This technique has been initiated in [P81a], [P81d] where it was used to prove that any separable II_1 factor M contains a MASA $A \subset M$ whose *normalizer* $\mathcal{N}_M(A) := \{u \in \mathcal{U}(M) \mid uAu^* = A\}$ generates a factor (A is *semiregular* in M ; see [P81a]), as well as a MASA $A \subset M$ whose normalizer is trivial, i.e. $\mathcal{N}_M(A) = \mathcal{U}(A)$ (A is *singular* in M ; see [P81d]).

In this paper we obtain more refined applications of this method, by combining it with two additional ingredients: the intertwining by bimodule technique ([P01], [P03]) and local properties of the ambient II_1 factor M , such as existence

Supported in part by NSF Grant DMS-1400208 and a Simons Fellowship

of non-trivial central sequences (i.e., property Gamma of [MvN43]) and *s-thin approximation*, a property that we introduce here and which will be defined shortly.

Recall in this respect that if Q, P are von Neumann subalgebras in a II_1 factor M , then we write $Q \prec_M P$ if there exists a Hilbert $Q-P$ sub-bimodule $\mathcal{H} \subset L^2 M$ such that $\dim \mathcal{H}_P < \infty$. In certain cases (notably if Q, P are MASAs) this condition is equivalent to the existence of a non-zero partial isometry $v \in M$ such that $v^*v \in Q$ and $vQv^* \subset P$.

Our first result shows that any separable II_1 factor M contains an uncountable family of singular (respectively semiregular) MASAs $\{A_i\}_i$ such that $A_i \not\prec_M A_j$, $\forall i \neq j$, with A containing non-trivial central sequences of M whenever M does. This will in fact follow from the following stronger result.

0.1. Theorem. *Let M be a separable II_1 factor M and $N \subset M$ a subfactor with trivial relative commutant, $N' \cap M = \mathbb{C}$. Let $P_n \subset M$ be a sequence of von Neumann subalgebras such that $N \not\prec_M P_n$, $\forall n$. Then N contains a singular (respectively semiregular) maximal abelian $*$ -subalgebra A of M such that $A \not\prec_M P_n$, $\forall n$. Moreover, if $N \simeq R$, then one can take A so that to satisfy $\mathcal{N}_M(A)'' = N$, and if N contains non-trivial central sequences of M , then A can be taken so that to contain non-trivial central sequences of M as well.*

We then consider the class of II_1 factors M which have an *s-MASA*, i.e., a MASA $A \subset M$ such that the von Neumann algebra $A \vee JAJ \subset \mathcal{B}(L^2 M)$, generated by left and right multiplication by elements in A on the Hilbert space $L^2 M$, is a MASA in $\mathcal{B}(L^2 M)$. We obtain a local characterization of factors in this class, by proving that M has an s-MASA if and only if it satisfies the following approximation property, that we call *s-thin*: for any finite partition $\{p_i\}_i \subset M$, any finite set $F \subset M$ and any $\varepsilon > 0$, there exist a partition $\{q_j\}_j \subset M$ refining $\{p_i\}_i$ and an element $\xi \in M$ such that any $x \in F$ can be ε -approximated in the norm- $\|\cdot\|_2$ by linear combinations of elements of the form $q_j \xi q_k$. We show that factors with s-MASAs are closed under amplifications and inductive limits and combine their local characterization with the iterative procedure to prove the following:

0.2. Theorem. *If M has an s-MASA, then there exist uncountably many non-intertwinable s-MASAs in M , which in addition can be chosen singular (resp. semiregular).*

The typical example of s-MASAs in II_1 factors are the *Cartan* (or *regular*) MASAs, i.e., MASAs $A \subset M$ for which $\mathcal{N}_M(A)'' = M$ (cf [FM77]). Any group measure space II_1 factor $M = L^\infty(X) \rtimes \Gamma$, obtained from a free ergodic measure preserving action $\Gamma \curvearrowright X$ of a countable group Γ on a probability measure space (X, μ) , has $A = L^\infty(X)$ as a Cartan subalgebra, which is thus also an s-MASA.

The above result shows that such factors necessarily have singular s-MASAs as well. Note that when M is hyperfinite, this fact was already known since ([D54], [Pu61]), where the first concrete examples of singular s-MASAs were given.

By [OP07], [PV11], [PV12], there are large classes of group measure space II_1 factors that have unique (up to unitary conjugacy) Cartan subalgebras (= regular MASAs), while by Theorem 0.2 above, such a factor always has “many” non conjugate semiregular s-MASAs.

There are by now several classes of II_1 factors known to have no Cartan subalgebras, obtained first by using free probability theory ([Vo96], then by using deformation-rigidity theory ([OP07], [CS11], [CSU11], [PV11], [PV12], [I12]). It is interesting to note that in each case when one could prove absence of Cartan MASAs by using free probability, the same techniques could be used to show absence of s-MASAs as well (notably for the free group factors $L(\mathbb{F}_n)$, cf. [G98]).

While there is much evidence that II_1 factors with s-MASAs but no Cartan subalgebras do exist, the problem of constructing such examples remains open. Another open problem is to find new proofs for the non-existence of s-MASAs in certain II_1 factors, such as the free group factors $L(\mathbb{F}_n)$. But perhaps the most “urgent” open problem in this direction is to find an intrinsic, local characterization of II_1 factors having Cartan subalgebras. Such an intrinsic characterization may lead to interesting applications in deformation-rigidity theory. It may also allow to prove that the class of factors with Cartan MASAs is close to inductive limits, a permanence property that, as we mentioned above, factors with s-MASAs do have. We discuss these open problems and other related questions in the last section of the paper.

This work has been finalized while I was visiting RIMS and Kyoto University in September 2016. I am very grateful to Masaki Izumi and Narutaka Ozawa for the warm hospitality extended to me during my stay.

Added in the proof. In their very recent paper *Thin II_1 factors with no Cartan subalgebras* (math.OA/1611.02138), Anna Krogager and Stefaan Vaes were able to construct a large class of II_1 factors that have s-MASAs but no Cartan subalgebras (in fact are even strongly solid) thus solving a problem stated above and in 5.1.2.

1. PRELIMINARIES

All finite von Neumann algebras that we consider in this paper will come with a fixed normal faithful trace state, denoted τ , and they will always be assumed separable with respect to the Hilbert norm $\|\cdot\|_2$ implemented by τ . If M is a finite von Neumann algebra and $B \subset M$ is a von Neumann subalgebra, then E_B denotes the unique τ -preserving conditional expectation of M onto B . If $B \subset M$ is merely

a weakly closed $*$ -subalgebra of M (so 1_B not necessarily equal to 1_M), then we use the same notation E_B for the unique trace preserving expectation of $1_B M 1_B$ onto B that preserves the trace state $\tau(\cdot)/\tau(p)$ on $1_B M 1_B$. We denote by $\mathcal{U}(M)$ the unitary group of a von Neumann algebra M . If \mathcal{X} is a Banach space and $S \subset \mathcal{X}$ is a subset, then we denote by $(S)_1$ the set of elements in S that have norm at most 1. For all notations that are not specified in the text, we send the reader to the expository notes ([P06]), and for basics on von Neumann algebras to the classic book [D57].

1.1. Perturbation of projections. The following result is well known, but we state it here in the specific form needed in this paper.

1.1.1. Lemma. *Let M be a finite von Neumann algebra, $B \subset M$ a diffuse von Neumann subalgebra and $e \in \mathcal{P}(M)$. Then there exists a projection $f \in B$ of trace equal to $\tau(e)$ such that $\|f - e\|_2 \leq 14\|e - E_B(e)\|_2 + \sqrt{13}\|e - E_B(e)\|_2$.*

Proof. By (Lemma 1.1 in [P81c]), if p is the spectral projection of $E_B(e)$ corresponding to the interval $[1/2, 1]$, then $\|p - E_B(e)\|_2 \leq 13\|e - E_B(e)\|_2$. Note that p also satisfies $\tau(p) \geq \tau(E_B(e)) = \tau(e)$.

By using the Cauchy-Schwartz inequality, this implies that $|\tau(p) - \tau(e)| \leq 13\|e - E_B(e)\|_2$. Thus, if $f \in p B p$ is a projection of trace $\tau(e)$, then $|\tau(f) - \tau(p)| \leq 13\|e - E_B(e)\|_2$. Altogether,

$$\begin{aligned} \|f - e\|_2 &\leq \|f - p\|_2 + \|p - E_B(e)\|_2 + \|E_B(e) - e\|_2 \\ &\leq 14\|e - E_B(e)\|_2 + \sqrt{13}\|e - E_B(e)\|_2. \end{aligned}$$

□

1.2. Embedding $L^\infty([0, 1])$ in II_1 factors. We will view a diffuse abelian von Neumann subalgebra A of a separable II_1 factor M as an embedding of $L^\infty([0, 1]) \simeq A$ into M . Thus, if $L^\infty([0, 1])$ is represented as an inductive limit of finer and finer partitions (e.g., dyadic) generating the σ -algebra of Lebesgue measurable subsets of $[0, 1]$, then such an embedding is determined by the corresponding increasing sequence of finite dimensional subalgebras $A_n \nearrow A$.

As pointed out in [P13], any embedding $L^\infty([0, 1]) \simeq A \subset M$ acts weak mixingly on $M \ominus A' \cap M$, and this entails the following 2-independence property:

1.2.1. Theorem [P13]. *Let M be a finite von Neumann algebra and $B \subset M$ a diffuse von Neumann subalgebra. Given any finite set $F \subset M \ominus B \vee (B' \cap M)$, any $n \geq 2$ and any $\varepsilon > 0$, there exists a partition of 1 with projections in B , $p_1, \dots, p_n \in \mathcal{P}(B)$ of trace $1/n$, such that $\|p_i x p_i\|_2^2 - \tau(p_i)^2 \tau(x^* x) \leq \varepsilon$, $\forall x \in F$.*

The fact that $L^\infty([0, 1]) \simeq A \subset M$ is a MASA is an extremality condition for the embedding, which can be described locally as follows (see e.g. [P81a]):

1.2.2. Lemma. *Let M be a separable finite von Neumann algebra and $B \subset M$ a von Neumann subalgebra. Let $A_n \subset B$ be an increasing sequence of finite dimensional von Neumann subalgebras and denote $A = \overline{\cup_n A_n}$. Let $\{x_j\}_j \subset M$ be a countable set, $\|\cdot\|_2$ -dense in the unit ball of M . Then A is maximal abelian in B if and only if $\lim_n \|E_{(A'_n \cap B) \vee B' \cap M}(x_j) - E_{A_n \vee B' \cap M}(x_j)\|_2 = 0$, $\forall j \geq 1$.*

Let us also mention a result from (A.1 in [P92]). We will derive it here from 1.2.1 and 1.2.2 above:

1.2.3. Corollary. *Let M be a separable finite von Neumann algebra and $B \subset M$ a diffuse von Neumann subalgebra. There exists an abelian von Neumann subalgebra $A \subset B$ such that $A' \cap M = A \vee B' \cap M$.*

Proof. Let $\{x_j\}_j \subset (M)_1$ be $\|\cdot\|_2$ -dense sequence in the unit ball of M . We construct recursively an increasing sequence of finite dimensional abelian von Neumann algebras $A_m \subset B$ such that $\|E_{A'_m \cap M}(x_j) - E_{A_m \vee B' \cap M}(x_j)\|_2 \leq 2^{-m}$ for all $1 \leq j \leq m$. Assuming we have constructed these algebras up to $m = n$, we construct A_{n+1} as follows. By Theorem 1.2.2, given any $\alpha > 0$, there exists an abelian finite dimensional *-subalgebra A_{n+1}^0 containing A_n such that $\|E_{A_{n+1}^0 \cap M}(x_j) - E_{B \vee B' \cap M}(x_j)\|_2 < \alpha$, $1 \leq j \leq n+1$. Then by taking first a MASA A^0 in B that contains A_{n+1}^0 and then using Lemma 1.2.1, we find a finite dimensional abelian subalgebra $A_{n+1}^\alpha \subset A^0$ that contains A_{n+1}^0 , such that $\|E_{(A_{n+1}^\alpha \cap B) \vee (B' \cap M)}(x_j) - E_{A_{n+1}^\alpha \vee B' \cap M}(x_j)\|_2 < \alpha$, $1 \leq j \leq n+1$. Taking α sufficiently small and letting $A_{n+1} = A_{n+1}^\alpha$, we get

$$\|E_{(A'_{n+1} \cap M)}(x_j) - E_{A_{n+1} \vee B' \cap M}(x_j)\|_2 < 2\alpha \leq 2^{-n-1}, 1 \leq j \leq n+1.$$

But then $A = \overline{\cup_n A_n}^w \subset B$ clearly satisfies the required condition. \square

1.3. Intertwining subalgebras in factors. We recall here some basic facts about the “intertwining” subordination relation between subalgebras in II_1 factors, from [P01], [P03]. We will follow the presentation [P05a] of this topic, which emphasized the “intertwining space” between subalgebras.

Thus, if M is a finite von Neumann algebra and $Q, P \subset M$ are weakly closed *-subalgebras of M , then $\mathcal{I}_M(Q, P)$ denotes the set of vectors $\xi \in L^2 M$ with the property that the Hilbert Q - P bimodule $\overline{\text{sp} Q \xi P} \subset L^2(M)$ has finite dimension as a right P -module. This space is clearly invariant to taking sums and to multiplication by Q from the left and P from the right. We call it the *intertwining Q - P subbimodule* of M .

The space $\mathcal{I}_M(Q, P)$ has left support $\leq 1_Q$ and right support $\leq 1_P$, it is invariant to multiplication from the left by $Q' \cap M$ and from the right by $P' \cap M$ and it is

increasing in P and decreasing in Q . Also, $\mathcal{I}_M(Q, P) = \mathcal{I}_M(Q_1, P_1)$ whenever $Q_1 \subset Q$, $P \subset P_1$ have finite index in the sense of [PP84] (either the “probabilistic” definition, or the existence of a finite orthonormal basis; see 1.2 in [P94] for the equivalence between these alternative definitions). Moreover, if $q \in Q$, $p \in P$ are projections that have central trace of support 1 in Q , respectively P , then $\mathcal{I}_M(qQq, pPp) = q\mathcal{I}_M(Q, P)p$.

We’ll denote as usual by $\langle M, P \rangle$ the basic construction algebra, defined as the commutant in $\mathcal{B}(L^2(M1_P))$ of the algebra of right multiplication by elements in P . It is also equal to the von Neumann algebra generated by operators of the form xe_Py^* , with $x, y \in M1_P$, acting on $\xi \in M1_P \subset L^2(M1_P)$ by $xe_Py^*(\xi) = xE_P(y^*\xi)$.

Then the projection $s_{Q,P} := \vee \{ \xi e_P \xi^* \mid \xi \in \mathcal{I}_M(Q, P) \}$ is equal to the support of the direct summand of $Q' \cap 1_Q \langle M, P \rangle 1_Q$ generated by projections that are finite in $1_Q \langle M, P \rangle 1_Q$. Thus, if $\xi \in L^2(M)$, then $\xi \perp \mathcal{I}_M(Q, P)$ iff $\xi e_P \xi^* s = 0$ and iff $\xi e_P \xi^*$ is orthogonal on any projection $q' \in Q' \cap \langle M, P \rangle$ with $q' \langle M, P \rangle q'$ finite.

If $\mathcal{I}_M(Q, P) \neq 0$, then we say that Q can be intertwined into P inside M , and write $Q \prec_M P$. Theorem 2.1 in [P03] shows that this condition is equivalent to the following: there exist projections $p \in P$, $q \in Q$, a unital isomorphism $\psi : qQq \rightarrow pPp$ (not necessarily onto) and a partial isometry $v \in M$ such that $vv^* \in (qQq)' \cap qMq$, $v^*v \in \psi(qQq)' \cap pMp$, $xv = v\psi(x)$, $\forall x \in qQq$, and $x \in qQq$, $xvv^* = 0$, implies $x = 0$. Justified by this 2nd characterization, one also uses the terminology *a corner of Q can be embedded into P inside M* (cf. 2.4 in [P03]).

By (2.1 in [P03]), the relation $Q \prec_M P$ is also equivalent to the fact that the action $\text{Ad}\mathcal{U}(Q)$ has a non-zero part that’s “compact relative to P ”. This means by definition that the commutant of Q in the semifinite von Neumann algebra $1_Q \langle M, P \rangle 1_Q$ contains non-zero finite projections or, equivalently, that the action $\text{Ad}\mathcal{U}(Q) \curvearrowright L^2(1_Q \langle M, P \rangle 1_Q, \text{Tr})$ has non-zero fixed points.

By (1.3 in [P01]), $Q \prec_M P$ is also equivalent to the fact that $Q' \cap 1_Q \langle M, P \rangle 1_Q$ contains non-zero elements from the ideal $\mathcal{J}(\langle M, P \rangle)$ of elements in $\langle M, P \rangle$ that are “compact relative to P ”.

We will use the notation $Q \not\prec_M P$ when the above conditions are not satisfied, i.e., when $\mathcal{I}_M(Q, P) = 0$. This means that the action Ad -action of $\mathcal{U}(Q)$ on $L^2(1_Q \langle M, P \rangle 1_Q, \text{Tr})$ is weak mixing. With the terminology (2.9 in [P05b]), in the case $1_Q = 1_M$ this amounts to $\text{Ad}\mathcal{U}(Q) \curvearrowright M$ being *weak mixing relative to P* .

We recall from (2.3 in [P03]) some useful necessary and sufficient criteria for the condition $Q \not\prec_M P$ to be satisfied.

1.3.1. Theorem. *Let M be a finite von Neumann algebra and $P, Q \subset M$ be weakly closed $*$ -subalgebras. For each $q \in \mathcal{P}(Q)$, fix $\mathcal{U}_q \subset \mathcal{U}(qQq)$ a subgroup generating qQq as a von Neumann algebra. The following conditions are equivalent:*

- (1) $Q \not\prec_M P$
- (2) *There exists a total subset $X \subset M$ and a sequence $u_n \in \mathcal{U}_1$ such that $\lim_n \|E_P(xu_ny)\|_2 = 0, \forall x, y \in X$.*
- (3) *Given any $q \in \mathcal{P}(Q)$ there exists a sequence of unitary elements $u_n \in \mathcal{U}_q$ such that $\lim_n \|E_P(xu_ny)\|_2 = 0, \forall x, y \in M$.*

Moreover, if P is regular in M , then the above are also equivalent to:

- (4) *There exists a total subset $X \subset M$ and a sequence $u_n \in \mathcal{U}_1$ such that $\lim_n \|E_P(xu_n)\|_2 = 0, \forall x \in X$.*

The proof of the above theorem in ([P03]) actually shows the following more general result, involving the intertwining space (cf. [P05a]):

1.3.2. Theorem. *With the same assumptions as in 1.3.1, if $X \subset L^2(1_Q M 1_P)$, then the following conditions are equivalent:*

- (1) $X \perp \mathcal{I}_M(Q, P)$.
- (2) *There exist $u_n \in \mathcal{U}_1$ such that $\lim_n \|E_P(\xi^* u_n \xi)\|_1 = 0, \forall \xi \in X$.*
- (3) *There exist $u_n \in \mathcal{U}_1$ such that $\lim_n \|E_P(\eta^* u_n \xi)\|_1 = 0, \forall \xi \in X, \eta \in L^2(M)$.*

1.3.3. Remarks. (a) Property 1.3.2(2) above, for characterizing the orthogonal in $L^2(M)$ of the intertwining space $\mathcal{I}_M(Q, P)$, can be traced back to ([P81b]), where this type of condition appears in the case of subalgebras $Q = L(G_1), P = L(G_2)$ of $M = L(G)$, arising from subgroups $G_1, G_2 \subset G$, as well as for general M and $Q = P$ (as in 1.4 below).

(b) The relation $Q \prec_M P$ is a “virtual” subordination relation, in the sense that it is “insensitive to finite index perturbations”: if Q or P are replaced by subalgebras $Q_1 \subset Q, P_1 \subset P$ of finite index (in the sense of one of the definitions in ([PP84])), then we still have $Q_1 \prec_M P_1$. In particular, if Q has a finite dimensional direct summand, then $Q \prec_M P$ for any $P \subset M$, and if there exist projections $p \in P, p' \in P' \cap M$ such that $pPpp' = pp'Mpp'$, then $Q \prec_M P$ for any $Q \subset M$. The relation $Q \prec_M P$ is also insensitive to localization to “corners” of the algebras involved, i.e., it is sufficient to be satisfied under non-zero projections of Q, P (or of their commutants in M).

(c) Related to (b) above, let us underline here that the notions of finite index (up to taking “corners”) for an inclusion of finite von Neumann algebras, $P \subset M$ considered in [PP84], generalizing the Jones index in the case of inclusions of factors [J83], translates into the relation $M \prec_M P$. More precisely, this last condition means that there exist projections $p \in P, p' \in P' \cap M$ such that $pPpp' = P_0 \subset M_0 = pp'Mpp'$ has finite index, either in the sense that there exists a finite orthonormal

basis of M_0 over P_0 ([PP84]) or that $E_{P_0}(x) \geq cx$, $\forall x \in (M_0)_+$, for some $c > 0$ (see A. 1 in [V07]).

In turn, the opposite relation $M \not\prec_M P$ translates into the fact that P has *uniform infinite index* in M and it amounts to $\mathcal{U}(pp'Mpp')$ containing sequences of elements that are “more and more” perpendicular to $pPpp'$, for any $p \in \mathcal{P}(P)$, $p' \in \mathcal{P}(P' \cap M)$. This type of condition characterizing infinite index can be traced back to (2.2 in [PP84]).

(d) Since it is determined by its behavior on corners, the subordination relation \prec_M is not transitive in general. For instance, if we take $Q = pMp + \mathbb{C}(1 - p)$, $P = \mathbb{C}p + \mathbb{C}(1 - p)$ then we have $M \prec_M Q$, $Q \prec_M P$, but $M \not\prec_M P$. For this same reason, requiring $Q \prec_M P$ and $P \prec_M Q$, does not define a “reasonable” equivalence relation \sim_M between subalgebras of M (e.g., the previous example would show that $M \sim \mathbb{C}$). However, for MASAs of M , we have the following (cf. A.1 in [P01]):

1.3.4. Theorem. *Let M be a finite von Neumann algebra and $A, B \subset M$ be MASAs in M . Then $A \prec_M B$ if and only if $B \prec_M A$ and if and only if there exists a non-zero partial isometry $v \in M$ such that $vv^* \in B$, $v^*v \in A$ and $vAv^* = Bvv^*$.*

1.4. Normalizing subalgebra. If M is a finite von Neumann algebra and $B \subset M$ is a von Neumann subalgebra, then we denote $\mathcal{N}_M(B) := \{u \in \mathcal{U}(M) \mid uBu^* = B\}$, the *normalizer* of B in M . The von Neumann algebra it generates, $\mathcal{N}_M(B)''$, is called the *normalizing von Neumann algebra* of B in M .

A von Neumann subalgebra B is *singular* in M if any automorphism $\text{Ad}(u)$ implemented by some $u \in \mathcal{N}_M(B)$ is inner, i.e., it is of the form $\text{Ad}(v)$ for some $v \in \mathcal{U}(B)$. This is the same as requiring that $\mathcal{N}_M(B) = \mathcal{U}(B)\mathcal{U}(B' \cap M)$. If in turn $\mathcal{N}_M(B)'' = M$, then we say that B is *regular* in M .

This terminology has been introduced in [D54], in the case $B = A \subset M$ is a maximal abelian *-subalgebra (MASA) in M . Note that for a MASA $A \subset M$, being singular means that $\mathcal{N}_M(A)'' = A$, or equivalently $\mathcal{N}_M(A) = \mathcal{U}(A)$, i.e., the normalizer of A in M acts trivially on A . A regular MASA will be called a *Cartan subalgebra* (or *Cartan MASA*) in M . We will also consider MASAs $A \subset M$ for which $\mathcal{N}_M(A)''$ is a factor (equivalently, $\mathcal{N}_M(A)$ acts ergodically on A), which will be called *semi-regular* (cf. [D54]).

More generally, recall from ([P97] and 1.4 in [P01]) that if $B \subset M$ is a von Neumann subalgebra then $q\mathcal{N}_M(B)$ denotes the set of all $x \in M$ with the property that there exists $x_1, \dots, x_n \in M$ such that $Bx \subset \sum_i x_i B$ and $xB \subset \sum_i Bx_i$. The space $q\mathcal{N}_M(B)$ is a *-subalgebra and we see that, by (Lemma 1.4.2 in [P01]), one has $q\mathcal{N}_M(B) = \mathcal{I}_M(B, B) \cap \mathcal{I}_M(B, B)^* \cap M$, with the weak closure being a von Neumann subalgebra of M .

Note that if $B = A$ is a MASA in M , then by (1.3 in [P01]) we have $q\mathcal{N}(A) = \text{sp}\mathcal{N}_M(A) = \mathcal{I}_M(A, A) \cap M$ and this space is $\|\cdot\|_2$ -dense in $\mathcal{I}_M(A, A)$. Thus, the normalizing von Neumann algebra of A satisfies $\mathcal{N}_M(A)'' = \overline{q\mathcal{N}_M(A)}^w$ and the orthogonal of this space in $L^2(M)$ coincides with $\mathcal{I}_M(A, A)^\perp$. So Theorem 1.3.2 entails the following criterion for estimating the size of the normalizer of A in M :

1.4.1. Corollary. *Let M be a finite von Neumann algebra, $A \subset M$ a MASA and $N = \mathcal{N}_M(A)''$ its normalizing von Neumann algebra. The following conditions are equivalent for an element $\xi \in L^2(M)$:*

- (1) $\xi \perp N$.
- (2) $\exists \{u_n\}_n \subset \mathcal{U}(A)$ such that $\lim_n \|E_A(\xi^* u_n \xi)\|_1 = 0$.
- (3) *For any $n \geq 2$ and any $\varepsilon > 0$, there exists an abelian von Neumann subalgebra $A_0 \subset A$ with n projections of equal trace such that $\|E_A(\xi^* y \xi)\|_1 \leq \varepsilon$, $\forall y \in (A_0 \ominus \mathbb{C}1)_1$.*

Proof. By applying 1.3.2 to the case $Q = P = A$, and taking into account that for a MASA $A \subset M$ one has $\mathcal{I}_M(A, A)^\perp = \mathcal{N}_M(A)^\perp$, it follows that (1) \Leftrightarrow (2). Then by taking $\mathcal{U}_1 \subset \mathcal{U}(A)$ to be a subgroup satisfying $\mathcal{U}_1'' = A$ and $\tau(u) = 0$, $u^2 = 1$, $\forall u \in \mathcal{U}_1 \setminus \{1\}$, we get (2) \Leftrightarrow (3). \square

Finally, let us note that a singular MASA $A \subset M$ means an embedding of the diffuse abelian von Neumann algebra $L^\infty([0, 1]) \simeq A \subset M$ so that the Ad-action of its unitary group on M is weak mixing relative to A , a Cartan MASA is an embedding so that this action is compact relative to A , while a semi-regular MASA is an embedding having a large relative compact part.

2. CONSTRUCTING MASAS WITH CONTROL OF INTERTWINERS

Results in [P81a], [P81d] show that any separable II_1 factor M has semi-regular and singular MASAs. The proof consists in constructing an embedding of $L^\infty([0, 1]) \simeq A \subset M$ as an inductive limit of dyadic partitions $A_n \nearrow A$ that become “more and more extremal in M ” (resulting into A being a MASA), while also controlling the normalizer of A , making it become singular (in [P81d]), respectively semi-regular (in [P81a]).

For A to become singular, one needs A_n to “become more and more relative weak mixing”. For it to become semi-regular, it is sufficient to build A_n so that fixed matrix units having A_n as diagonal are in the normalizer (i.e., in the relative compact part), at each step n .

We will show below how one can use much more of the intertwining by bimodules criteria within such iterative procedure, allowing us to construct embeddings

$L^\infty([0, 1]) \simeq A \subset M$ so that to be weak mixing relative to a given countable family of subalgebras of M . One can in fact even control such relative weak mixingness when M is embedded into larger II_1 factors, thus leading to super-rigidity type properties for A . Moreover, we will do the construction so that to also take into account local properties of the ambient factor M , such as existence of central sequences (in this section), and s-thin approximation (in the next section).

2.1. Theorem. *Let N be a separable II_1 factor and $N \hookrightarrow M_n$ be embeddings of N into separable II_1 factors such that $N' \cap M_n$ is of type I, $\forall n$. Let also $P_n \subset M_n$ be von Neumann subalgebras.*

1° *There exists a MASA $A \subset N$ such that for each n one has:*

- (a) $\mathcal{N}_{M_n}(A) = \mathcal{U}(A \vee N' \cap M_n)$;
- (b) $\mathcal{I}_{M_n}(A, P_n)^\perp = \mathcal{I}_{M_n}(N, P_n)^\perp$;
- (c) $M'_n \cap A^\omega$ is non-trivial whenever $M'_n \cap N^\omega$ is non-trivial.

In particular, A is singular in N and if $N' \cap M_n = \mathbb{C}1$, then A is a singular MASA in M_n which contains non-trivial central sequences of M_n whenever N does.

2° *There exists a semiregular MASA $A \subset N$ such that for each n one has:*

- (a) $A' \cap M_n = A \vee N' \cap M_n$;
- (b) $\mathcal{N}_{M_n}(A)'' \subset N \vee N' \cap M_n$;
- (c) $\mathcal{I}_{M_n}(A, P_n)^\perp = \mathcal{I}_{M_n}(N, P_n)^\perp$.
- (d) $M'_n \cap A^\omega$ is non-trivial whenever $M'_n \cap N^\omega$ is non-trivial.

Moreover, if $N \simeq R$ then one can take $A \subset N$ such that $\mathcal{N}_{M_n}(A)'' = N \vee N' \cap M_n$.

Proof. For each M_n choose a sequence $\{x_k^n\}_k \subset (M_n)_1$ that's $\|\cdot\|_2$ -dense in $(M_n)_1$ and a sequence $\{\xi_k^n\}_k \subset L^2(M_n) \ominus \mathcal{I}_{M_n}(N, P_n)$ that's $\|\cdot\|_2$ -dense in $L^2(M_n) \ominus \mathcal{I}_{M_n}(N, P_n)$.

Let also $\{e_m\}_m \subset \{e \in \mathcal{P}(N) \mid \tau(e) \leq 1/2\}$ be a $\|\cdot\|_2$ -dense sequence.

To prove 1°, we construct recursively a sequence of finite dimensional abelian von Neumann subalgebras $A_m \subset N$ together with projections $f_m \in \mathcal{P}(A_m)$, with $\tau(f_m) = \tau(e_m)$, and unitary elements $v_m \in \mathcal{U}(A_m f_m)$, $w_m, u_m \in \mathcal{U}(A_m)$, satisfying the following properties for all $1 \leq i, j, k \leq m$:

$$(2.1.1) \quad \|f_m - e_m\|_2 \leq 13\|e_m - E_{A'_{m-1} \cap N}(e_m)\|_2$$

$$(2.1.2) \quad \|E_{A'_m \cap M_k}(x_i^{k*} v_m x_j^k)(1 - f_m)\|_2 \leq 2^{-m},$$

$$(2.1.3) \quad \|E_{A'_m \cap M_k}(x_j^k) - E_{A_m \vee N' \cap M_k}(x_j^k)\|_2 \leq 2^{-m},$$

$$(2.1.4) \quad \|E_{P_k}(x_i^{k*} w_m \xi_j^k)\|_2 \leq 2^{-m},$$

$$(2.1.5) \quad \|[x_i^k, u_m]\|_2 \leq 2^{-m}, \|E_{A_{m-1}}(u_m)\|_2 \leq 2^{-m}.$$

Assume we have constructed $(A_m, f_m, v_m, w_m, u_m)$ satisfying these properties for $m = 1, 2, \dots, n$. By applying Lemma 1.1.1 to $B = A'_n \cap N$ and $e = e_{n+1}$, it follows that there exists $f_{n+1} \in A'_n \cap N$ such that $\|f_{n+1} - e_{n+1}\|_2 \leq 13\|e_{n+1} - E_{A'_n \cap N}(e_{n+1})\|_2$. By Corollary 1.2.3, there exists a MASA $B_0 \in (1 - f_{n+1})N(1 - f_{n+1})$ satisfying the property

$$B'_0 \cap (1 - f_{n+1})M_k(1 - f_{n+1}) = B_0 \vee (N' \cap M_k), \forall 1 \leq k \leq n+1.$$

Since $(A_n f_{n+1})' \cap f_{n+1} N f_{n+1}$ is type II_1 and $B_0 \vee (N' \cap M_k)(1 - f_{n+1})$ are of type I, for each k we have $(A_n f_{n+1})' \cap f_{n+1} N f_{n+1} \not\prec_{M_k} B \vee (N' \cap M_k)(1 - f_{n+1})$. Thus, there exists $v_{n+1} \in \mathcal{U}((A_n f_{n+1})' \cap f_{n+1} N f_{n+1})$ with the property that

$$(2.1.6) \quad \|E_{B'_0 \cap M_k}((1 - f_{n+1})x_i^{k*} v_{n+1} x_j^k (1 - f_{n+1}))\|_2 < 2^{-n-1}, 1 \leq i, j, k \leq n+1.$$

Moreover, we may clearly assume v_{n+1} has finite spectrum. We then take a refinement A_{n+1}^0 of A_n in N that contains f_{n+1} , such that $A_{n+1}^0 f_{n+1}$ contains v_{n+1} , while $A_{n+1}^0(1 - f_{n+1})$ “approximates” B_0 well enough (in the sense of Lemma 1.2.1) so that, due to (2.1.6) and its strict inequality, we still have

$$(2.1.7) \quad \|E_{A_{n+1}^0 \cap M_k}((1 - f_{n+1})x_i^{k*} v_{n+1} x_j^k (1 - f_{n+1}))\|_2 < 2^{-n-1}, 1 \leq i, j, k \leq n+1.$$

On the other hand, by Corollary 1.2.3, there exists a finite dimensional abelian von Neumann subalgebra A_{n+1}^1 in N that contains A_{n+1}^0 and satisfies

$$(2.1.8) \quad \|E_{A_{n+1}^1 \cap M_k}(x_j^k) - E_{A_{n+1}^1 \vee N' \cap M_k}(x_j^k)\|_2 \leq 2^{-n-1}, 1 \leq i, j, k \leq n+1.$$

Now, since $A_{n+1}^1 \cap N$ has finite index in N , by Section 1.3 we have $\mathcal{I}_{M_k}(A_{n+1}^1 \cap N, P_k) = \mathcal{I}_{M_k}(N, M_k)$, and thus $\xi_j^k \perp \mathcal{I}_{M_k}((A_{n+1}^1 \cap N, P_k)$, for all $1 \leq j, k \leq n+1$. Thus, by Theorem 1.3.2, there exists a unitary element $w_{n+1} \in (A_{n+1}^1 \cap N)$ such that

$$(2.1.9) \quad \|E_{P_k}(x_i^{k*} w_{n+1} \xi_j^k)\|_2 < 2^{-n-1}, 1 \leq i, j, k \leq n+1.$$

Also, we may clearly assume w_{n+1} has finite spectrum. We take $A_{n+1}^2 \subset N$ to be the finite dimensional abelian von Neumann algebra generated by A_{n+1}^1 and w_{n+1} . Due to (2.1.7), (2.1.8) and (2.1.9), A_{n+1}^2 satisfies conditions (2.1.1) – (2.1.4).

Finally, by using the fact that $M'_n \cap N^\omega \neq \mathbb{C}$ implies $M'_n \cap N^\omega$ diffuse (because M_n is a factor), it follows that for any $\alpha > 0$ there exists a projection $p \in N$ of trace $1/2$ such that $\|[x, p]\|_2 < \alpha$ for all $x \in \{x_i^k \mid 1 \leq i, k \leq n+1\} \cup (A_{n+1}^2)_1$. By taking α sufficiently small and using Lemma 1.1.1, it follows that there exists $u_{n+1} \in \mathcal{U}(A_{n+1}' \cap N)$ sufficiently close to $1 - 2p$ so that we have $\|[x_i^k, u_{n+1}]\|_2 \leq 2^{-n-1}$, for all $1 \leq i, k \leq n+1$. Thus, if we define $A_{n+1} = A_{n+1}^2 \vee \{u_{n+1}\}''$, then all conditions (2.1.1) – (2.1.5).

Define $A = \overline{\bigcup_n A_n}^w$. Condition (2.1.3) clearly implies $A' \cap M_k = A \vee N' \cap M_k$, $\forall k$, while condition (2.1.4) implies $A \not\prec_{M_k} P_k$, $\forall k$.

Let $u \in \mathcal{N}_{M_m}(A)$. If $\text{Ad}(u)$ acts non-trivially on A , then there exists a non-zero projection $e \in A$ of trace $\leq 1/2$ such that $u^*eu \leq 1 - e$. Let n_0 be large enough so that $2^{-n_0} < \|e\|_2/8$. Since $\{e_n\}_n$ is $\|\cdot\|_2$ -dense in the set of projections of N of trace $\leq 1/2$, there exists $n \geq n_0$ such that $\|e_n - e\|_2 < \|e\|_2/8$ and such that there exists $j, k \leq n$ with $\|x_j^k - eu\|_2 < \|e\|_2/8$. But then, if $f_n, v_n \in A_n \subset A$ are as given by the construction, we would have $u^*ev_nu \in A \subset A'_n \cap M_m$ as well as the estimates

$$\begin{aligned}
 (2.1.10) \quad \|ef_n\|_2 &= \|ev_n\|_2 = \|E_{A'_n \cap M_m}(u^*ev_nu)(1 - e)\|_2 \\
 &\leq \|E_{A'_n \cap M_m}(u^*v_nu)(1 - f_n)\|_2 + \|e\|_2/8 \\
 &\leq \|E_{A'_n \cap M_m}(x_j^{k*}v_nx_j^k)(1 - f_n)\|_2 + 3\|e\|_2/8.
 \end{aligned}$$

But since $\|e_n - e\|_2 < \|e\|_2/8$, we also have

$$\|f_ne - e\|_2 \leq \|f_n - e\|_2 \leq \|f_n - e_n\|_2 + \|e_n - e\|_2 \leq 2\|e_n - e\|_2 < \|e\|_2/4,$$

which together with (2.1.10) and (2.1.2) implies that

$$\|e\|_2/2 > \|ef_n\|_2 \geq \|e\|_2 - \|e\|_2/4 = 3\|e\|_2/4,$$

a contradiction. This shows that $\mathcal{N}_{M_k}(A) = \mathcal{U}(A' \cap M_k) = \mathcal{U}(A \vee N' \cap M_k)$, $\forall k$, finishing the proof that A satisfies all the conditions in part 1° of the theorem.

To prove 2°, note first that by Corollary 1.2.3 there exists a hyperfine II_1 subfactor $R \subset N$ such that $R' \cap M_k = N' \cap M_k$, $\forall k$. It is then sufficient to construct a Cartan subalgebra A of R such that $A' \cap M_k = A \vee N' \cap M_k$, $\mathcal{N}_{M_k}(A)'' = R \vee N' \cap M_k$, $\mathcal{I}_{M_n}(A, P_n)^\perp = \mathcal{I}_{M_n}(N, P_n)^\perp$, $\forall k$. In other words, it is sufficient to prove the last part of 2°, where one assumes $N \simeq R$ and want to construct a Cartan MASA $A \subset N$ whose normalizing algebra is exactly $N \vee N' \cap M_k$.

To this end, we construct recursively a sequence of commuting dyadic matrix subalgebras $R_m \subset N$ (i.e., $R_m \simeq M_{2^{k_m} \times 2^{k_m}}(\mathbb{C})$, for some $k_m \geq 1$), with diagonal

subalgebras $D_m \subset R_m$, such that if we denote $N_m = \vee_{k=1}^m R_k$, $A_m = \vee_{k=1}^m D_k$, there exist a projection f_m of trace $1/2$ in D_m and unitary elements $v_m \in \mathcal{U}(D_m f_m)$, $w_m \in \mathcal{U}(D_m)$, so that if we denote $y_i^k = x_i^k - E_{N \vee N' \cap M_k}(x_i^k)$, then the following properties are satisfied for $1 \leq i, j, k \leq m$:

$$(2.1.11) \quad \|E_{A'_m \cap M_k}(y_i^{k*} v_m y_j^k)(1 - f_m)\|_2 \leq 1/10;$$

$$(2.1.12) \quad \|f_m(y_i^k)(1 - f_m)\|_2 \geq 2\|y_i^k\|_2/5;$$

$$(2.1.13) \quad \|E_{A'_m \cap M_k}(x_j^k) - E_{A_m \vee N' \cap M_k}(x_j^k)\|_2 \leq 2^{-m};$$

$$(2.1.14) \quad \|E_{P_k}(x_i^{k*} w_m \xi_j^k)\|_2 \leq 2^{-m};$$

$$(2.1.15) \quad \|E_{N_m}(x_i^k) - E_N(x_i^k)\|_2 \leq 2^{-m}.$$

Assuming we have constructed these objects up to $m = n$, we construct them for $m = n + 1$ as follows.

Noticing that the finite set $F = \{y_j^k \mid 1 \leq j, k \leq n + 1\}$ is perpendicular to $N_n \vee N' \cap M_k$ (which is equal to the commutant in M_k of $N'_n \cap N$), by Lemma 1.2.1 we can first pick a projection f of trace $1/2$ in $N'_n \cap N$ such that f is almost 2-independent to F . In particular, we can choose f so that for all $1 \leq j, k \leq n + 1$ we have

$$(2.1.16) \quad \|f y_j^k (1 - f)\|_2 \geq 2\|y_j^k\|_2/5.$$

By Corollary 1.2.2, there exists a MASA B_0 in $(1 - f)(N'_n \cap N)(1 - f)$ such that $B'_0 \cap (1 - f)M_k(1 - f) = B_0 \vee (N_n \vee N' \cap M_k)(1 - f)$, $\forall 1 \leq k \leq n + 1$. Since $f(N'_n \cap N)f$ is type II_1 and $B_0 \vee (N_n \vee N' \cap M_k)(1 - f)$ is type I , the former cannot be intertwined into the latter inside M_k , so by Theorem 1.3.2 there exists a unitary element $v \in f(N'_n \cap N)f$ such that for all $1 \leq i, j, k \leq n + 1$ we have

$$(2.1.17) \quad \|E_{B'_0 \cap (1-f)M_k(1-f)}((1-f)x_i^{k*} v x_j^k(1-f))\|_2 < 1/10.$$

Moreover, we may choose v so that to belong to a dyadic finite dimensional abelian subalgebra $B_1^1 \subset f(N'_n \cap N)f$. Also, by approximating B_0 sufficiently well

with a dyadic finite dimensional subalgebra $B_1^0 \subset B_0$, we will still have for all $1 \leq i, j, k \leq n+1$ the estimates

$$(2.1.18) \quad \|E_{B_1^{0'} \cap (1-f)M_k(1-f)}((1-f)x_i^{k*} v x_j^k (1-f))\|_2 < 1/10.$$

We now take $B_1 \subset N'_n \cap N$ to be a dyadic finite dimensional abelian subalgebra containing $B_1^1 f + B_1^0(1-f)$. Since $\xi_j^k \perp \mathcal{I}_{M_k}(N, P_k) = \mathcal{I}_{M_k}(B_1' \cap N, P_k)$, there exists a unitary element $w \in B_1' \cap N$ such that

$$(2.1.19) \quad \|E_{P_k}(x_i^{k*} w x_j^k)\|_2 \leq 2^{-n-1}, 1 \leq i, j, k \leq n+1,$$

We may clearly also assume w lies in a dyadic finite dimensional abelian subalgebra $B_2 \subset N'_n \cap N$ that contains B_1 . Take now $R_{n+1}^0 \subset N'_n \cap N$ to be a (dyadic) finite dimensional factor having B_2 as a diagonal algebra. Finally, since $N \simeq R$, there exists a dyadic finite dimensional factor $R_{n+1} \subset N'_n \cap N$ that contains R_{n+1}^0 , such that if we define $N_{n+1} = N_n \vee R_{n+1}$ then

$$(2.1.20) \quad \|E_{N_{n+1}}(x_i^k) - E_N(x_i^k)\|_2 \leq 2^{-n-1}, 1 \leq i, k \leq n+1.$$

Thus, if we take D_{n+1} to be a diagonal of R_{n+1} that contains B_2 and denote $A_{n+1} = A_n \vee D_{n+1}$, $N_{n+1} = N_n \vee R_{n+1}$, $f_{n+1} = f$, $v_{n+1} = v$, $w_{n+1} = w$, then (2.1.16) – (2.1.20) insure that conditions (2.1.11) – (2.1.15) are satisfied for $n+1$.

Let now $R = \vee_k R_k = \overline{\cup_n N_n^w}$, $A = \vee_k D_k = \overline{\cup_n A_n^w}$. Condition (2.1.15) clearly implies that $R = N$, while condition (2.1.13) implies $A' \cap M_k = A \vee N' \cap M_k$ and (2.1.14) implies $A \not\prec_{M_k} P_k, \forall k$.

By construction, we have that A is Cartan in N , so that $\mathcal{N}_{M_k}(A)''$ contains $N \vee N' \cap M_k$. If this inclusion is strict for some k , then by the factoriality of N there must exist an automorphism θ of A implemented by a unitary $u \in \mathcal{N}_{M_k}(A)$ such that $\theta \circ \text{Ad}(v)$ acts freely on A , $\forall v \in \mathcal{N}_N(A)$. Thus, $E_{N \vee N' \cap M_k}(u) = 0$.

Let $x_j^k \in M_k$ be so that $\|x_j^k - u\|_2 \leq 1/20$. This implies that $\|y_j^k - u\|_2 \leq 1/20$ and that for each $n \geq k$ we have $\|y_j^{k*} v_n y_j^k - u^* v_n u\|_2 \leq 1/10$, while by (2.1.12) we also have

$$\|f_n u(1 - f_n)\|_2 \geq \|f_n y_j^k(1 - f_n)\|_2 - 1/10 \geq 2\|y_j^k\|_2/5 - 1/10$$

Thus, since $u^* v_n u \in A$, by (2.1.11) we get

$$\begin{aligned} 1/10 &\geq \|E_{A \vee N' \cap M_k}(y_i^{k*} v_n y_j^k)(1 - f_n)\|_2 \\ &\geq \|(1 - f_n)u^* v_n u(1 - f_n)\|_2 - 1/10 = \|f_n u(1 - f_n)\|_2 - 1/10 \end{aligned}$$

$$\geq 2\|y_j^k\|_2/5 - 1/5 \geq 2/5 - 1/20 - 1/5 = 3/20,$$

which is a contradiction. \square

Recall that in [P81d] one proves existence of singular MASAs not only in II_1 factors, but also in II_∞ and III_λ factors, for $0 < \lambda < 1$. This result is obtained as a consequence of stronger statement about MASAs in a (separable) II_1 factor M , showing that given any group of automorphisms \mathcal{G} of the associated II_∞ factor $M^\infty = M \overline{\otimes} \mathcal{B}(\ell^2\mathbb{N})$ such that $\mathcal{G}/\text{Int}(M^\infty)$ is countable, there exists a \mathcal{G} -singular MASA $A \subset M$, i.e., a maximal abelian subalgebra of M with the property that if $\theta \in \mathcal{G}$ satisfies $\theta(a) \in A$, for all a in a “corner” Ap of A , then θ acts as the identity on Ap . Let us note here that this type of result is in fact covered by the above general theorem:

2.2. Corollary. *Let M be a II_1 factor with a sequence of von Neumann subalgebras of uniform infinite index $P_n \subset M$ (in the sense of 1.3.3.(c)). Let also $\mathcal{G} \subset \text{Aut}(M^\infty)$ be a subgroup of automorphisms of M^∞ that contains $\text{Int}(M^\infty)$ and is so that $\mathcal{G}/\text{Int}(M)$ is countable. Then there exists a \mathcal{G} -singular MASA $A \subset M$ such that $A \not\prec_M P_n$, $\forall n$. In particular, M contains uncountably many non-intertwinable \mathcal{G} -singular MASAs, which in addition can be taken to contain non-trivial central sequences of M whenever M has property Gamma.*

Proof. Let $\theta_n \in \mathcal{G}$ be a sequence of automorphisms of M^∞ such that $\mathcal{G} = \cup_n \theta_n \circ \text{Int}(M^\infty)$. For each n , let s_n be so that $\text{Tr} \circ \theta_n = s_n \text{Tr}$ and denote $t_n = 1 + s_n$. Let $M_n = M^{t_n}$ and $f_n \in M_n$ a projection of trace $\tau(f_n) = 1/(1 + s_n)$. Thus, $f_n M_n f_n \simeq M$ and we can embed M into M_n as the subfactor $\{x \oplus \theta_n(x) \mid x \in M \simeq f_n M_n f_n\}$.

By part 1° of Theorem 2.1, there exists a singular MASA $A \subset M$ such that $A' \cap M_n = Af_n + A(1 - f_n)$ and $\mathcal{N}_{M_n}(A) = \mathcal{U}(Af_n + A(1 - f_n))$. It is immediate to see that this means A is \mathcal{G} -singular in M . Moreover, by 2.1.1° we can take A so that to also satisfy $A \not\prec_{M_n} P_n$ (when P_n is viewed as a subalgebra of M_n). This of course implies $A \not\prec_M P_n$ as well.

To prove the last part, let \mathcal{F} be a maximal family of \mathcal{G} -singular MASAs of M such that A, B are not intertwinable for any $A \neq B$ in \mathcal{F} . Assume \mathcal{F} is countable and note that $M \not\prec_{M_n} A$, $\forall A \in \mathcal{F}$. By applying Theorem 2.1 to $N = M \subset M_n$ (with M_n as above), $\forall n$, and $\{P_n\}_n = \mathcal{F}$, it follows that there exists a singular MASA $C \subset M$ such that $C \not\prec_M A$, $\forall A \in \mathcal{F}$, contradicting the maximality of \mathcal{F} . \square

2.3. Corollary. *Any separable II_1 factor contains an uncountable family of mutually non-conjugate semi-regular MASAs, which in addition can be chosen to contain non-trivial central sequences if the ambient factor has property Gamma.*

Proof. The same argument as in the above proof applies, using 2.1.2° instead of 2.1.1°. \square

It has been shown in (2.5 of [P81b]) that if $P \subset M$ is a von Neumann subalgebra and $u \in \mathcal{U}(M)$ satisfies the property that for all $n \geq 1$ and all $\varepsilon > 0$ there exists a subalgebra $A_0 \simeq L(\mathbb{Z}/n\mathbb{Z}) \subset Q$ with $u^* A_0 u \perp_\varepsilon P$, then $u \perp \mathcal{N}_M(P)$. Along these lines, one can now deduce the following stronger result (generalizing 1.4.1 as well):

2.4. Corollary. *Let M be a finite von Neumann algebra. Let $Q, P \subset M$ be diffuse von Neumann subalgebras and $\xi \in L^2(M)$. The following conditions are equivalent:*

1° $\xi \perp \mathcal{I}_M(Q, P)$.

2° For any $n \geq 1$ and any $\varepsilon > 0$ there exists $u \in \mathcal{U}(Q)$ such that $u^n = 1$, $\tau(u^k) = 0$, $1 \leq k < n$ and $\|E_P(\xi^* u^k \xi)\|_1 \leq \varepsilon$, $1 \leq k < n$.

3° For any $n \geq 1$ and any $\varepsilon > 0$, there exists an n -dimensional abelian von Neumann subalgebra $A_0 \subset Q$ such that $\tau(p) = 1/n$ for any minimal projection in A_0 and $\|E_P(\xi^* x \xi)\|_1 \leq \varepsilon$ for all $x \in (A_0)_1$ with $\tau(x) = 0$.

Proof. We clearly have $3^\circ \Leftrightarrow 2^\circ$ and by Theorem 1.3.2 we have $2^\circ \Rightarrow 1^\circ$.

To prove $1^\circ \Rightarrow 3^\circ$, let $\xi \perp \mathcal{I}_M(Q, P)^\perp$. Apply first Theorem 2.1 to get a MASA $A \subset Q$ with the property that $\mathcal{I}_M(A, P)^\perp = \mathcal{I}_M(Q, P)^\perp$. Representing A as the von Neumann algebra of the countable group $\mathbb{Z}/2\mathbb{Z}^{\oplus \infty}$, we get a unitary group $\mathcal{U}_1 \subset \mathcal{U}(A)$ such that $\mathcal{U}_1'' = A$, $u^2 = 1$, $\tau(u) = 0$, $\forall u \in \mathcal{U}_1 \setminus \{1\}$. Applying Theorem 1.3.2 to $\xi \perp \mathcal{I}_M(A, P)$, we get a sequence $\{u_m\}_m \subset \mathcal{U}_1$ such that $\lim_m \|E_P(\xi^* u_m \xi)\|_1 = 0$. Taking $\alpha > 0$ appropriately small and A_0 to be an n -dimensional subalgebra of A generated by projections of trace $1/n$ that's α -contained in $\{u_m \mid m_0 \leq m \leq m_1\}''$, for some $m_0 \leq m_1$ sufficiently large so that $\|E_P(\xi^* u_m \xi)\|_1 < \alpha$, $\forall m \geq m_0$, one gets condition 3° satisfied. \square

3. THIN FACTORS AND MASAS WITH BOUNDED MULTIPLICITY

In the 1950s, W. Ambrose and I.M. Singer have considered MASAs in II_1 factors $A \subset M$ with the property that the von Neumann algebra $A \vee JAJ \subset \mathcal{B}(L^2 M)$, generated by the left-right multiplication on $L^2 M$ by elements in A , is maximal abelian in $\mathcal{B}(L^2 M)$. Noticing that this is equivalent to $A \vee JAJ$ having a cyclic vector (i.e., $\exists \xi \in L^2 M$ with $[A\xi A] = L^2 M$), this property is analogue to an inclusion of groups $H \subset G$ with just one (non-trivial) double co-set over H . (Note however that the algebra framework makes it so that regular MASAs do satisfy this property (cf. [FM77]), while for a normal subgroup $H \subset G$, $H \setminus G/H = G/H$ is always large.)

In [Pu61] L. Pukanszky took this idea further, by noticing that the type of the algebra $(A \vee JAJ)' \subset \mathcal{B}(L^2 M)$ is an invariant for the isomorphism class of a MASA inclusion $A \subset M$, and that if an inclusion of groups $H \subset G$ with G ICC and H abelian is so that $H \setminus G/H = H$ has n identical classes, then the MASA

inclusion $A = L(H) \subset L(G) = M$ has the property that $A \vee JAJ$ has multiplicity n on $L^2(M) \ominus L^2A$. In other words, the commutant algebra $(A \vee JAJ)'$, which is always equal to $Ae_A \simeq A$ on the reducing space L^2A , is homogeneous of type I_n on $L^2(M \ominus A)$. Taking appropriate examples $H \subset G$ with G locally finite ICC (inspired by a construction in [D54]), he was able to give examples of singular MASAs in the hyperfinite II_1 factor R that have “multiplicity n ”, for each $1 \leq n < \infty$, and are thus mutually non-conjugate by automorphisms of R .

The type of the von Neumann algebra $(A \vee JAJ)'(1 - e_A) \subset \mathcal{B}(L^2(M \ominus A))$ (i.e., the list of multiplicities appearing in its decomposition as a direct sum of homogeneous type I_{n_i} algebras, $1 \leq n_i \leq \infty$) is what one generically calls the *Pukanszky invariant* of $A \subset M$.

Of this, we will retain here only the supremum over all the multiplicities $1 \leq n_i \leq \infty$ in the decomposition $(A \vee JAJ)' = \oplus_i L^\infty(X_i) \otimes M_{n_i \times n_i}(\mathbb{C})$.

3.1. Definitions. 1° Let M be a II_1 factor and $A \subset M$ an abelian von Neumann subalgebra. We denote by $m(A \subset M)$ the supremum over all $1 \leq m \leq \infty$ with the property that $(A \vee JAJ)'$ has a type I_m direct summand, and call it the *multiplicity* of $A \subset M$. Notice that $m(A \subset M) = 1$ if and only if $A \vee JAJ$ is maximal abelian in $\mathcal{B}(L^2M)$, and that this implies A is maximal abelian in M (see 3.3 below). An abelian von Neumann subalgebra A in M with the property that $A \vee JAJ$ is maximal abelian in $\mathcal{B}(L^2M)$ is called an *s-MASA* of M .

2° If M is a II_1 factor then we denote $m_a(M) = \min\{m(A \subset M) \mid A \text{ a MASA in } M\}$. Thus, $m_a(M) = 1$ if and only if M has an s-MASA. A II_1 factor with this property is called an *s-thin* factor.

Let us note right away that the multiplicity of MASAs behaves well to taking tensor products and intermediate subfactors: if $A_i \subset M_i$, $i = 1, 2$, are inclusions of MASAs, then $m(A_1 \overline{\otimes} A_2 \subset M_1 \overline{\otimes} M_2) = m(A_1 \subset M_1)m(A_2 \subset M_2)$; also, if $B \subset Q \subset P$ is a MASA in P with Q an intermediate factor, then $m(B \subset Q) \leq m(B \subset P)$.

3.2. Examples. By (2.9 in [FM77]), any Cartan MASA in a II_1 factor is an s-MASA. But by [Pu61], there do exist singular s-MASAs as well. For instance, when $M \simeq R$ is the hyperfine II_1 factor, then besides its (unique by [CFW81]) Cartan subalgebra D , R contains an s-MASA A that is singular. More precisely, the following example of a singular MASA $A \subset R$ from ([D54]) has been shown in ([Pu61]) to be an s-MASA: represent R as the group factor associated with the amenable ICC group G of affine transformation on \mathbb{Q} , with its abelian subgroups $T = \mathbb{Q}$ (translations), $H = \mathbb{Q}^*$ (homotheties); since G is ICC relative to both T and H , $D = L(T)$, $A = L(H)$ are MASAs in $R = L(G)$; since T is normal in G , D is a Cartan subalgebra in R , while since H acts transitively on $T \setminus \{0\}$, A follows singular in R , with the vector $\xi = \xi_0 + \xi_1 \in \ell^2(G) = L^2R$ cyclic for $A \vee JAJ$, where

ξ_0 is the vector corresponding to the trivial element in $T \subset G$ (so translation by 0) and ξ_1 is the element in $T \subset G$ corresponding to translation by 1.

3.3. Remark. As we mentioned in 3.1.1° above, if A is a von Neumann subalgebra of M , then the condition “ $A \vee J_M A J_M$ maximal abelian in $\mathcal{B}(L^2 M)$ ” implies that A is a MASA in M . To see this, note first that this condition implies $B = A' \cap M$ abelian. Indeed, since $e_B \in (A \vee J_M A J_M)' = A \vee J_M A J_M$, it follows that $e_B(A \vee J_M A J_M)e_B$ is maximal abelian in $\mathcal{B}(L^2 B)$. Since by the commutativity of B we have $A = J_B A J_B$, this forces $A = B$.

The terminology “s-MASA in M ” can thus be viewed as emphasizing a *strengthening* of the property of being a MASA in M . The prefix “s” can also be viewed as hinting to the terminology *simple MASA*, which has been sometimes used for abelian von Neumann subalgebras satisfying this property (N.B.: this has been the original Ambrose-Singer terminology, carried on in [K67], [JP82], [Ge97]). The usage of the adjective “simple” for a MASA can however be misleading, as (non-trivial) abelian von Neumann algebras do have non-trivial ideals and are thus not simple as rings... The terminology “simple MASA” may trigger additional confusion as it has also been used by Takesaki in [T63], but for a different class of MASAs $A \subset M$, via a characterization which has in fact been later shown equivalent to A being singular (cf. [H79]).

From this point on, if S is a non-empty subset of a Hilbert space \mathcal{H} , we will use the notation $[S]$ for the Hilbert subspace generated by S , i.e., $[S] = \overline{\text{sp}(S)}$, and also for the orthogonal projection of \mathcal{H} onto this space, the difference being always clear from the context.

The following result provides alternative characterizations of MASA-multiplicity.

3.4. Proposition. *Let M be a separable II_1 factor, $A \subset M$ an abelian von Neumann subalgebra and $n_0 \geq 1$ an integer. The following conditions are equivalent:*

- 1° *Any type I_m direct summand of $(A \vee J A J)'$ satisfies $m \leq n_0$.*
- 2° *There exists $X \subset L^2 M$ with $|X| \leq n_0$ such that $[A X A] = L^2 M$.*
- 3° *$\forall F \subset M$ finite, $\forall \delta > 0$, there exists $X \subset L^2 M$ such that $|X| \leq n_0$ and $F \subset_\delta \text{sp} A X A$.*
- 4° *The representation $\text{Ad}(\mathcal{U}(A)) \curvearrowright L^2(M \ominus A' \cap M)$ admits a cyclic set of n_0 vectors.*

Moreover, if $n_0 = 1$, then in 2° above one can take $X = \{b\}$ with $b = b^ \in M$.*

Before proving this result, let us notice the following:

3.5. Lemma. *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be an abelian von Neumann algebra acting on the Hilbert space \mathcal{H} .*

1° The supremum over all $n \leq \infty$ such that \mathcal{A}' has a I_n direct summand is equal to the minimum over all $m \leq \infty$ for which there exists $X \subset \mathcal{H}$ with $|X| = m$ and $[\mathcal{A}X] = \mathcal{H}$.

2° For any $\eta_1, \eta_2 \in \mathcal{H}$, the set $L = \{t \in \mathbb{C} \setminus \{0\} \mid [\mathcal{A}'(\eta_1 + t\eta_2)] \neq [\mathcal{A}'(\eta_1)] \vee [\mathcal{A}'(\eta_2)]\}$ is at most countable.

Proof. 1° This part of the statement is the case “ B abelian” of the Murray-von Neumann coupling constant theorem (see [vN43]) relating a finite von Neumann algebra $B \subset \mathcal{B}(\mathcal{H})$ with its commutant $B' \subset \mathcal{B}(\mathcal{H})$, by the “factor of multiplicity” $\dim_B \mathcal{H}$.

2° This part is just (Lemma 3.5 in [P82]). □

Proof of 3.4. Denote $\mathcal{A} = A \vee JAJ$ and $\mathcal{B} = (A \vee JAJ)' \cap \mathcal{B}(L^2M)$. By 3.5.1° in the above Lemma, we have $1^\circ \Leftrightarrow 2^\circ$, and we clearly have $2^\circ \Rightarrow 3^\circ$.

Condition 3° shows that there exists a sequence of subsets $X_n \subset L^2M$ of cardinality at most n_0 such that if we denote $p_n = [AX_nA] \in \mathcal{B}$ then $p_n \rightarrow 1$ in the so -topology. By Lemma 3.5.1°, each $p_n \mathcal{B} p_n$ is of finite type with all homogeneous type I_m summands satisfying $m \leq n_0$. By ([PP84], or 1.2 in [P94]), this is equivalent to having the (probabilistic) Pimsner-Popa index of $\mathcal{A}p_n \subset p_n \mathcal{B} p_n$ at most equal to n_0 . Since the definition of this index behaves well to limits (see e.g., [PP84]), it follows that the index of $\mathcal{A} \subset \mathcal{B}$ is $\leq n_0$ as well, which in turn means that any type I_m direct summand of \mathcal{B} must satisfy $m \leq n_0$. Thus, $3^\circ \Rightarrow 1^\circ$.

To prove $4^\circ \Leftrightarrow 2^\circ$, we need to show that there exists a set $X \subset L^2(M \ominus A)$ with $|X| \leq n_0$, such that $\text{sp}\{u\xi u^* \mid u \in \mathcal{U}(A), \xi \in X\}$ is dense in $L^2(M \ominus A)$. But by the “linearization principle” (5.1 in [P89]), a set $X \subset L^2(M \ominus A' \cap M)$ is cyclic for $\text{Ad}(\mathcal{U}(A)) \curvearrowright L^2(M \ominus A' \cap M)$ if and only $[AXA] = L^2(M \ominus A' \cap M)$.

To prove the last part of the statement, assume there exists $\xi \in L^2M$ such that $[A\xi A] = L^2(M)$. We need to show there exists $b = b^*$ such that $[AbA] = L^2M$.

Let us first note that there exists $\eta_0 = \eta_0^* \in L^2(M)$ such that $[A\eta_0 A] = [A\xi A] = L^2M$. Indeed, this follows by noticing that $\mathcal{A} = A \vee JAJ$ satisfies $\mathcal{A}' = \mathcal{A}$ and so one can apply Lemma 3.5.2° to $\eta_1 = \Re \xi$, $\eta_2 = \Im \xi$, $\mathcal{A} \subset \mathcal{B}(L^2M)$, to get some $t \in \mathbb{R}$ such that $\eta_0 = \eta_1 + t\eta_2$ satisfies $[\mathcal{A}(\eta_0)] = [\mathcal{A}(\eta_1)] \vee [\mathcal{A}(\eta_2)] = [\mathcal{A}(\xi)]$.

Denote $y_n = e_{[-n, -n+1)}(\eta_0)\eta_0 + e_{[n-1, n)}(\eta_0)\eta_0$, $n \geq 1$, and note that $y_n = y_n^* \in M$ are mutually orthogonal in L^2M with $\|y_n\| \leq 2n$, and that $\eta_0 = \lim_m \sum_{k=1}^m y_k$ in L^2M .

By applying recursively 3.5.2°, we get scalars $0 \leq t_m \leq 1/2m$ such that $b_m = \sum_{k=1}^m t_k y_k / 2^k$ satisfy $\{y_1, \dots, y_m\} \subset \text{sp}Ab_m A$. Denote $b = \lim_m b_m \in (M_h)_1$. Since $\text{sp}A\eta_0 A$ is $\|\cdot\|_2$ -dense in L^2M and η_0 is the $\|\cdot\|_2$ -limit of $z_m = \sum_{k=1}^m y_k$, it follows that for any $F \subset [A\xi_0 A]$ finite and any $\alpha > 0$, there exists m_0 such that $F \subset_\alpha \text{sp}Az_{m_0} A$,

for any $m \geq m_0$. But $Az_m A \subset \Sigma_{k=1}^m \text{sp} A y_k A \subset \text{sp} A b_m A$. Thus, we also get $F \subset_\alpha \text{sp} A b_m A$, $\forall m \geq m_0$. But this implies $F \subset_\alpha \text{sp} A b A$ as well. Since F and α were arbitrary, this shows that $\text{sp} A b A$ is dense in $[A \xi_0 A] = L^2 M$. \square

3.6. Theorem. *Let M be a separable II_1 factor and $n_0 \geq 1$ an integer. The following conditions are equivalent:*

1° $m_a(M) \leq n_0$.

2° *For any finite dimensional abelian von Neumann subalgebra $A_0 \subset M$, any finite subset $F \subset M$ and any $\delta > 0$, there exists a finite dimensional abelian von Neumann algebra $A_1 \subset M$ containing A_0 and $X_1 \subset L^2 M$ with $|X_1| \leq n_0$ such that $F \subset_\delta \text{sp} A_1 X_1 A_1$.*

3° *There exists a sequence of positive numbers $t_n \searrow 0$ such that each II_1 factor $N = M^{t_n}$ satisfies the the following property:*

(3.6.3) *For any $F \subset N$ finite and $\delta > 0$, there exist $A_1 \subset N$ finite dimensional abelian and $X_1 \subset L^2 N$, with $|X_1| \leq n_0$, such that $F \subset_\delta \text{sp} A_1 X_1 A_1$.*

Proof. We clearly have $1^\circ \Rightarrow 2^\circ$. By applying property 2° to $A_0 = \mathbb{C}e + \mathbb{C}(1 - e)$, for projections e in M of trace $\tau(e) = t_n$, we see that $2^\circ \Rightarrow 3^\circ$.

To prove $2^\circ \Rightarrow 1^\circ$, let $\{x_n\}_n \subset (M)_1$ be a sequence of elements that's $\|\cdot\|_2$ -dense in $(M)_1$. We first construct recursively an increasing sequence of finite dimensional abelian von Neumann sbalgebras $A_m \subset M$ together with a sequence of subsets $X_m \subset L^2 M$, with $|X_m| \leq n_0$, such that $\{x_1, \dots, x_m\} \subset_{2^{-m}} \text{sp} A_m X_m A_m$.

Assume (A_m, X_m) have been constructed for $1 \leq m \leq n$. Apply 2° to $F = \{x_1, \dots, x_{n+1}\}$, $B_0 = A_n$ and $\delta = 2^{-n-1}$ to get a larger finite dimensional algebra $B_1 \supset B_0$, with a subset $X_{n+1} \subset L^2 M$ having at most n_0 elements, such that if we let $A_{n+1} = B_1$, then $\{x_1, \dots, x_{n+1}\} \subset_{2^{-n-1}} \text{sp} A_{n+1} X_{n+1} A_{n+1}$.

Let now $A = \overline{\cup_n A_n}^w$. By construction, it follows that $\forall F \subset M$ finite and $\varepsilon > 0$, there exists $X \subset L^2 M$, with $|X| \leq n_0$, such that $F \subset_\varepsilon \text{sp} A X A$. But then 3.4.4° above implies that $m(A \subset M) \leq n_0$.

Let us finally prove that $3^\circ \Rightarrow 2^\circ$. Let $F \subset (M)_1$ be a finite set, $B_0 \subset M$ a finite dimensional abelian von Neumann subalgebra and $\delta > 0$. We need to prove that there exists a larger finite dimensional abelian algebra $B_1 \supset B_0$ with a set $X_1 \in L^2 M$ of at most n_0 vectors such that $F \subset_\delta \text{sp} B_1 X_1 B_1$. It is clearly sufficient to prove this for B_0 generated by minimal projections $\{q_i \mid 0 \leq i \leq m\}$ with $\tau(q_i) = t_n$, $1 \leq i \leq m$ and $\tau(q_0) < t_n$, where n is sufficiently large so that $t_n < \delta/100$. Let v_1, \dots, v_m be partial isometries in M such that $v_i^* v_i = q_1$ and $v_i v_i^* = q_i$, $1 \leq i \leq m$.

Denote $F' = \{v_i^* y v_j \mid y \in F, 1 \leq i, j \leq n\} \subset q_1 M q_1$. By property 3°, there exists a finite dimensional abelian von Neumann algebra $A_1 \subset q_1 M q_1$ and a subset $X_1 \in q_1 L^2(M) q_1$ of at most n_0 elements, such that $F' \subset_{\delta/\sqrt{m}} \text{sp} A_1 X_1 A_1$. For each $\xi_1 \in X_1$, denote $\eta_1 = \sum_{i,j=1}^m v_i \xi_1 v_j^*$. Let Y_1 be the set of all such η_1 and denote $B_1 = \mathbb{C} q_0 + \sum_{i=1}^m v_i A_1 v_i^*$. Fix $y \in F$. Since $v_i^* y v_j \in F'$, there exist $a_{ij} \in \text{sp} A_1 X_1 A_1$ such that $\|v_i^* y v_j - a_{ij}\|_{2, q_1 M q_1}^2 \leq \delta^2/m$.

Note that, by the definitions of B_1 and η_1 , the element $b = \sum_{i,j=1}^m v_i a_{ij} v_j^*$ belongs to $\text{sp} B_1 Y_1 B_1$. Moreover, by using Pythagoras theorem in M , we have

$$\begin{aligned} \|y - b\|_2^2 &= \|y - \sum_{i,j=1}^m v_i a_{ij} v_j^*\|_2^2 = \sum_{i,j=1}^m \|v_i^* y v_j - a_{ij}\|_2^2 \\ &= \sum_{i,j=1}^m \|v_i^* y v_j - a_{ij}\|_{2, q_1 M q_1}^2 \tau(q_1) \leq m^2 \tau(q_1) \delta^2/m = \delta^2 \tau(1 - q_0). \end{aligned}$$

This shows that $F \subset_\delta \text{sp} B_1 Y_1 B_1$, thus finishing the proof. \square

Motivated by property (3.6.3) above, we will also consider the following:

3.7. Definition. Let M be a II_1 factor. We denote by $\text{wm}_a(M)$ the minimum over all cardinalities $1 \leq m \leq \infty$ with the property that given any finite set $F \subset M$ and any $\varepsilon > 0$, there exist a subset $X_1 \subset L^2 M$ with $|X_1| \leq m$ and a finite dimensional abelian $*$ -subalgebra $B_1 \subset M$ such that $F \subset_\varepsilon [B_1 X_1 B_1]$. The II_1 factor M is *weak s -thin* if $\text{wm}_a(M) = 1$, i.e., if for any finite set $F \subset M$ and any $\varepsilon > 0$, there exists a finite dimensional abelian von Neumann subalgebra $B_1 \subset M$ and a vector $\eta_1 \in L^2 M$ such that $F \subset_\varepsilon \text{sp} B_1 \eta_1 B_1$.

Note that $\text{m}_a(M)$, $\text{wm}_a(M)$ are both isomorphism invariants for M , that measure the “thinness” of M relative to its abelian subalgebras and satisfy $\text{m}_a(M) \geq \text{wm}_a(M)$. Notice also that the invariant $\text{m}_a(M)$ is very much in the spirit of what was called n -weak thinness in [GP99], which denoted the minimal cardinality $1 \leq n \leq \infty$ with the property that there exist hyperfinite von Neumann subalgebras $R_0, R_1 \subset M$ and a set $X \subset L^2(M)$ with $|X| \leq n$ such that $[R_0 X R_1] = L^2(M)$. More precisely, the minimal such n obviously satisfies $n \leq \text{m}_a(M)$.

3.8. Corollary. 1° We have $\text{m}_a(M) = \text{m}_a(M^t)$, for any $t > 0$. In particular, if M is an s -thin factor, then its amplifications M^t are s -thin factors, $\forall t > 0$. Also, if M is weak s -thin, then $M_{n \times n}(M)$ is weak s -thin, $\forall n \geq 1$.

2° If a II_1 factor M has non-trivial fundamental group (e.g., if M is a McDuff factor), then $\text{wm}_a(M) = \text{m}_a(M)$. In particular, such a II_1 factor is s -thin iff it is weak s -thin.

3° If a II_1 factor M is generated by an increasing sequence of subfactors $M_n \subset M$, then $\text{m}_a(M) \leq \limsup_n \text{m}_a(M_n)$ and $\text{wm}_a(M) \leq \limsup_n \text{wm}_a(M_n)$. In particular, if all M_n are s -thin (resp. weak s -thin), then M is s -thin (resp. weak s -thin).

Proof. To prove 1° , note first that if we assume $m_a(M) \leq n_0$, then by $1^\circ \Rightarrow 3^\circ$ in Proposition 3.6 there exists $t_n \searrow 0$ such that $\text{wm}_a(M^{t_n}) \leq n_0$. But then $\text{wm}_a((M^t)^{t_n/t}) \leq n_0$, with $t_n/t \searrow 0$, and thus by applying $3^\circ \Rightarrow 1^\circ$ in 3.6, it follows that $m_a(M^t) \leq n_0$.

Part 2° is immediate from the equivalence $1^\circ \Leftrightarrow 3^\circ$ in Proposition 3.6.

Part 3° can be easily deduced from the characterizations in Proposition 3.6. To see this, assume first that $m_a(M_n) \leq n_0$, $\forall n$. We will directly construct from this a MASA A in M such that $m_a(M) \leq n_0$.

Let $\{x_k\}_k$ be a sequence of elements in the unit ball of $\cup_n M_n$ which is $\|\cdot\|_2$ -dense in $(M)_1$. Assume we have constructed finite dimensional abelian von Neumann subalgebras $A_1 \subset A_2 \dots \subset A_m$ in the $*$ -algebra $\cup_n M_n$ together with subsets $X_1, \dots, X_m \subset \cup_n M_n$, with $|X_i| \leq n_0$, such that $\{x_1, \dots, x_k\} \subset_{2^{-k}} \text{sp} A_k X_k A_k$, $1 \leq k \leq m$. Let $K \geq 1$ be large enough such that $\{x_1, \dots, x_{m+1}\} \subset M_K$ and $A_m \subset M_K$. By applying 3.6.2 $^\circ$ to $B_0 = A_m$, $F = \{x_1, \dots, x_{m+1}\}$ and $\delta = 2^{-m-1}$, as well as the fact that $m_a(M_K) \leq n_0$, we get a larger finite dimensional abelian von Neumann subalgebra $A_{m+1} \supset A_m$ in M_K (which can thus be viewed as a subalgebra in $M \supset M_K$), with a set of $\leq n_0$ vectors $X_{m+1} \subset L^2(M_K) \subset L^2(M)$ with at most n_0 elements, such that $\{x_1, \dots, x_{m+1}\} \subset_{2^{-m-1}} \text{sp} A_{m+1} X_{m+1} A_{m+1}$.

Letting now $A = \overline{\cup_m A_m}^w$, it is trivial to see that $A \subset M$ satisfies condition 5 $^\circ$ in Proposition 3.4, and thus satisfies $m(A \subset M) \leq n_0$, implying that $m_a(M) \leq n_0$.

Assume now that $\text{wm}_a(M_n) \leq n_0$, $\forall n$. To prove that $\text{wm}_a(M) \leq n_0$, it is clearly sufficient to show that 3.7 holds true for any finite subset F in a prescribed $\|\cdot\|_2$ -dense subset $\mathcal{X} \subset (M)_1$. But then the inequality is trivial for $F \subset (\cup_n M_n)_1$. \square

4. SINGULAR AND SEMIREGULAR s-MASAs IN s-THIN FACTORS

We prove in this section that if M has an s-MASAs (i.e. M is s-thin), then it has singular and semi-regular s-MASAs (in fact, many of them). In other words, if M admits cyclic MASA actions of $L^\infty([0, 1])$, then it has cyclic MASA actions that are relative weak mixing, respectively have a “large” relative compact part.

4.1. Theorem. *Let N be a separable s-thin factor and $N \hookrightarrow M_n$ be embeddings of N into separable II_1 factors such that $N' \cap M_n$ is of type I, $\forall n$. For each n , let $P_n \subset M_n$ be a von Neumann subalgebra such that $N \not\prec_{M_n} P_n$.*

1 $^\circ$ There exists a singular s-MASA $A \subset N$ such that $\mathcal{N}_{M_n}(A)'' = A \vee N' \cap M_n$ and $A \not\prec_{M_n} P_n$, $\forall n$.

2 $^\circ$ There exists a semiregular s-MASA $A \subset N$ such that $\mathcal{N}_{M_n}(A)'' \subset N \vee N' \cap M_n$ and $A \not\prec_{M_n} P_n$, $\forall n$.

Moreover, in both 1° and 2°, if $M'_n \cap N^\omega \neq \mathbb{C}$ then one can choose A so that $M'_n \cap A^\omega \neq \mathbb{C}$ as well.

Proof. We proceed exactly as in the proof of Theorem 2.1, constructing A iteratively, but with an additional “local requirement” which will insure that in the end, besides being singular (resp. semiregular) in N , A is an s-MASA in N , with its Ad-action on M_n being weak mixing relative to P_n , $\forall n$.

Thus, we take $\{e_m\}_m \subset \{e \in \mathcal{P}(N) \mid \tau(e) \leq 1/2\}$ to be a $\|\cdot\|_2$ -dense sequence. Also, we let $\{x_k\}_k \subset (N)_1$ be $\|\cdot\|_2$ -dense in $(N)_1$ and for each M_n we choose a sequence $\{x_k^n\}_k \subset (M_n)_1$ that's $\|\cdot\|_2$ -dense in $(M_n)_1$.

To prove Part 1°, we construct recursively an increasing sequence of finite dimensional abelian von Neumann subalgebras $A_m \subset N$ together with projections $f_m \in \mathcal{P}(A_m)$ and unitary elements $v_m \in \mathcal{U}(A_m f_m)$, $w_m \in \mathcal{U}(A_m)$, as well as a vector $\xi_m \in L^2 N$, such that for each $1 \leq i, j, k \leq m$ we have:

$$(4.1.1) \quad \|f_m - e_m\|_2 \leq 13\|e_m - E_{A'_{m-1} \cap N}(e_m)\|_2$$

$$(4.1.2) \quad \|E_{A'_m \cap M_k}(x_i^{k*} v_m x_j^k)(1 - f_m)\|_2 \leq 2^{-m}, 1 \leq i, j, k \leq m$$

$$(4.1.3) \quad \|E_{A'_m \cap M_k}(x_j^k) - E_{A_m \vee N' \cap M_k}(x_j^k)\|_2 \leq 2^{-m},$$

$$(4.1.4) \quad \|E_{P_k}(x_i^{k*} w_m x_j^k)\|_2 \leq 2^{-m}, 1 \leq i, j, k \leq m.$$

$$(4.1.5) \quad \{x_1, \dots, x_m\} \subset_{2^{-m}} \text{sp} A_m \xi_m A_m.$$

Assume we have constructed $(A_m, f_m, v_m, w_m, \xi_m)$ satisfying these properties for $m = 1, 2, \dots, n$. By the proof of Theorem 2.1, we can first construct a finite dimensional abelian algebra $A_{n+1}^1 \subset M$ that contains A_n , with a projection $f_{n+1} \in A_{n+1}^1$ and unitary elements $v_{n+1} \in A_{n+1}^1 f_{n+1}$, $w_{n+1} \in A_{n+1}^1$, such that conditions (4.1.1) – (4.1.4) are satisfied for $m = n + 1$ and with A_{n+1}^1 playing the role of A_{n+1} . Finally, since A_{n+1}^1 is contained in the s-thin factor N , by 3.6.2° there exists a refinement $A_{n+1} \subset N$ of A_{n+1}^1 and a vector $\xi_{n+1} \in L^2 N$ such that condition (4.1.5) is satisfied for $m = n + 1$. Note that since A_{n+1} contains A_{n+1}^1 , conditions (4.1.1) – (4.1.4) will be satisfied for $m = n + 1$.

If we now denote $A = \overline{\cup_n A_n}^w$, then the same argument as in the proof of Theorem 2.1 shows that due to conditions (4.1.1) – (4.1.4) we have $A' \cap M_k =$

$A \vee N' \cap M_k$, $A \not\prec_{M_k} P_k$ and $\mathcal{N}_{M_k}(A) = \mathcal{U}(A \vee N' \cap M_k)$, while condition (4.1.5) implies $A \subset N$ satisfies condition 3.4.4° with $n_0 = 1$ and is thus an s-MASA in N .

In turn, to prove part 2° we construct recursively a sequence of commuting dyadic matrix subalgebras $R_m \subset N$ (i.e., $R_m \simeq M_{2^{k_m} \times 2^{k_m}}(\mathbb{C})$, for some $k_m \geq 1$), with diagonal subalgebras $D_m \subset R_m$, such that if we denote $N_m = \vee_{k=1}^m R_k$, $A_m = \vee_{k=1}^m D_k$, there exist a projection f_m of trace 1/2 in D_m , unitary elements $v_m \in \mathcal{U}(D_m f_m)$, $w_m \in \mathcal{U}(D_m)$, and a vector $\xi_m \in L^2 N$, such that if we denote $y_i^k = x_i^k - E_{N \vee N' \cap M_k}(x_i^k)$, then the following properties are satisfied for $1 \leq i, j, k \leq m$:

$$(4.1.6) \quad \|E_{A'_m \cap M_k}(y_i^{k*} v_m y_j^k)(1 - f_m)\|_2 \leq 1/10;$$

$$(4.1.7) \quad \|f_m y_i^k(1 - f_m)\|_2 \geq 2\|y_i^k\|_2/5;$$

$$(4.1.8) \quad \|E_{A'_m \cap M_k}(x_j^k) - E_{A_m \vee N' \cap M_k}(x_j^k)\|_2 \leq 2^{-m};$$

$$(4.1.9) \quad \|E_{P_k}(x_i^{k*} w_m x_j^k)\|_2 \leq 2^{-m};$$

$$(4.1.10) \quad \{x_1, \dots, x_m\} \subset_{2^{-m}} \text{sp} A_m \xi_m A_m.$$

Assuming we have constructed these objects up to $m = n$, we construct them for $m = n + 1$ as follows. From the proof of 2.1.2°, we can first construct a dyadic matrix algebra R_{n+1}^1 that commutes with N_n , together with a diagonal subalgebra $D_{n+1}^1 \subset R_{n+1}^1$, a projection $f_{n+1} \in D_{n+1}^1$ of trace 1/2 and unitaries $v_{n+1} \in D_{n+1}^1 f_{n+1}$, $w_{n+1} \in D_{n+1}^1$, such that if we denote $A_{n+1}^1 = A_n \vee D_{n+1}^1$, $N_{n+1}^1 = N_n \vee R_{n+1}^1$, then conditions (4.1.6) – (4.1.9) are satisfied for $m = n + 1$, with $A_{n+1}^1 \subset N_{n+1}^1$ in the role of $A_{n+1} \subset N_{n+1}$.

Let $\{e_{ij}\}_{i,j \in J}$ be matrix units for N_{n+1}^1 , with e_{ii} generating A_n , and denote $F = \{e_{1i} x_k e_{j1} \mid 1 \leq k \leq n + 1, i, j \in J\}$. Since $N_0 = (N_{n+1}^1)' \cap N$ is s-thin (as an amplification of N), it follows that there exists an abelian finite dimensional von Neumann subalgebra $B_0 \subset N_0$ and a vector $\eta_0 \in L^2(N_0)$, such that $F \subset_\alpha \text{sp} B_0 \eta_0 B_0$, where $\alpha = 2^{-n-1}/|J|$. Moreover, we may clearly also assume that B_0 is dyadic. If we now denote $B_1 = B_0 \vee A_n \in N$ and $\eta_1 = \sum_{i,j} e_{i1} \eta_0 e_{1j} \in L^2(N)$, then Pythagoras Theorem implies that $\{x_1, \dots, x_{n+1}\} \subset_{2^{-n-1}} \text{sp} B_1 \eta_1 B_1$. Finally, we take a dyadic matrix algebra $R_{n+1}^0 \subset N_0$ having B_0 as a diagonal subalgebra

and denote $R_{n+1} = R_{n+1}^1 \vee R_{n+1}^0$, $D_{n+1} = D_{n+1}^1 \vee B_0$, $N_{n+1} = N_n \vee R_{n+1}$, $A_{n+1} = B_1 = A_n \vee D_{n+1}$, $\xi_{n+1} = \eta_1$. It is then immediate to see that all conditions (4.1.6) – (4.1.10) are satisfied for $m = n + 1$.

Let $A = \overline{\cup_n A_n}^w = \vee_n D_n$, $R = \overline{\cup_n N_n}^w = \vee_n R_n$. Like in the proof of 2.1.2°, condition (4.1.8) insures that $A' \cap M_k = A \vee N' \cap M_k$ while (4.1.9) implies that $A \not\leq_{M_k} P_k$, $\forall k$. Also, by the definitions of A and R we see that $\mathcal{N}_{M_k}(A)'' \supset R \vee N' \cap M_k$, and thus A is semiregular in N . On the other hand, condition (4.1.10) shows that $\mathcal{N}_{M_k}(A)'' \subset N \vee N' \cap M_k$. Finally, condition (4.1.10) combined with the case $n_0 = 1$ of 3.4.4° show that A is s-MASA in N .

Finally, let us note that in the proof of the existence of singular s-MASAs in 1° (resp. of semi-regular s-MASAs in 2°), if we assume $M'_n \cap N^\omega \neq \mathbb{C}$, then exactly as in the proof of Theorem 2.1.1° (respectively 2.1.2°), one can complement the list of conditions in the recursive construction with a condition insuring that A contains non-trivial central sequences of M_n . \square

4.2. Corollary. *Let M be a separable s -thin II_1 factor. Then M has uncountably many mutually non-intertwinable singular (respectively semiregular) s -MASAs. Moreover, if M has the property Γ , then all these MASAs can be taken to contain non-trivial central sequences of M .*

Proof. The argument in the proofs of Corollaries 2.2 and 2.3 works exactly the same way, by using Theorem 4.1 in lieu of Theorem 2.1. \square

5. FINAL REMARKS AND OPEN PROBLEMS

5.1. Absence of Cartan MASAs versus s-MASAs. The first examples of (separable) II_1 factors without Cartan subalgebras were obtained by Voiculescu in [Vo96], who used free probability methods to prove that the free group factors $L(\mathbb{F}_n)$ do not have Cartan MASAs. It was then realized that a suitable adaptation of the argument in [Vo96] shows that $L(\mathbb{F}_n)$ doesn't have s-MASAs ([Ge98]), nor even MASAs with finite multiplicity ([GP99]), in fact $m_a(L(\mathbb{F}_n)) = \infty$. Similar arguments can be used to show that any II_1 factor of the form $M = N_1 * N_2$, with N_1, N_2 finitely generated diffuse von Neumann subalgebras of R^ω , satisfies $m_a(M) = \infty$. Indeed¹, this follows by combining (4.1 in [GP99]) with the lower estimates on free entropy dimension in [Ju01] and the additivity of Voiculescu's free entropy dimension ([Vo96]).

On the other hand, during the last ten years, a large number of results about absence of Cartan MASAs have been obtained through deformation rigidity theory ([OP07], [CS11], [CSU11], [PV11], [PV12], [I12]). For instance, it was shown in

¹I am grateful to Dima Shlyakhtenko for pointing out to me this line of arguments.

[PV11] that $L(\mathbb{F}_n) \overline{\otimes} N$ has no Cartan subalgebras for any finite factor N . In many “absence of Cartan MASA” results that are obtained through deformation-rigidity theory, one actually obtains classes of II_1 factors M that are *strongly solid* in the sense of [OP07], i.e., the normalizing algebra of any diffuse amenable $B \subset M$ (in particular of any MASA $A \subset M$) is amenable. This is notably the case for the free group factors $M = L(\mathbb{F}_n)$ ([OP07]) and more generally for all factors $L(\Gamma)$ arising from non-elementary hyperbolic groups Γ ([CS11]).

Notice that no result about automatic amenability of normalizing algebras of MASAs could be obtained using free probability, while absence of s-MASAs could not be shown by using deformation rigidity theory!

Finally, let us point out that absence of Cartan MASAs in a II_1 factor M amounts to having no relative compact actions by MASAs on M , while strong solidity means the relative compact part of any such action is amenable. Also, absence of s-MASAs in M means there are no cyclic actions by MASAs on M .

5.1.1. Problem. It would be interesting to find new proofs of absence of s-MASAs in certain factors. This is particularly the case for the II_1 factor $L(\mathbb{F}_n)$, where a direct, “elementary” proof seems possible.

5.1.2. Problem. We have no examples of II_1 factors with s-MASAs but without Cartan subalgebras. One class of factors that may provide such examples are the crossed product factors of the form $M = R \rtimes \Gamma$, with $\Gamma = \Gamma_1 \times \Gamma_2$ where Γ_1, Γ_2 are groups in one of the classes in [PV11], [PV12], for which one knows that any regular MASA of M is necessarily contained in R (after conjugacy by a unitary). Thus, if the Γ -action on R “mixes well” the Cartan MASAs of R , then one should be able to show that R cannot contain regular MASAs of M .

5.1.3. Problem. Is the weak s-thin property equivalent to s-thin? More generally, do we always have $\text{wm}_a(M) = \text{m}_a(M)$? We saw that once a factor M has non-trivial fundamental group, then the two properties are equivalent, but it is not clear whether this is the case for any II_1 factor.

5.1.4. Problem. Another question we leave open is whether $\text{m}_a(M) < \infty$ implies $\text{m}_a(M) = 1$ and whether there are permanence properties relating $\text{m}_a(M)$ with the multiplicity invariant $\text{m}_a(N)$ of its subfactors of finite Jones index $N \subset M$. In particular, whether M is s-thin if and only if N is s-thin.

5.2. Local characterization of factors with Cartan MASAs. As we have seen above, existence of Cartan MASAs is a property of II_1 factors, that many II_1 factors, such as $L(\mathbb{F}_n)$, do not have. Factors with Cartan MASAs are precisely the ones that admit relative compact actions of $L^\infty([0, 1])$. Let us call \mathcal{CF} the class of such II_1 factors. If A is a MASA in a II_1 factor M , then there are ways

to characterize the regularity property $\mathcal{N}_M(A)'' = M$ which does not specifically mention the normalizer of A . Thus, it is shown in [PS01] that A is Cartan in M iff there exists $\mathcal{U}_0 = \mathcal{U}_0^* \subset \mathcal{U}(M)$ such that $\overline{\text{sp}\mathcal{U}_0} = M$ and $A \ni a \mapsto E_A(uau^*) \in A$ are c.p. maps with discrete (countable) Fubini decomposition, $\forall u \in \mathcal{U}_0$, and also iff this is true for $\mathcal{U}_0 = \mathcal{U}$. It is also shown in [PV14] that A is Cartan iff $\mathcal{M} = \langle M, A \rangle$ has the Kadison-Singer norm-paving property relative to $\mathcal{A} = A \vee JAJ$ and iff there exists a normal conditional expectation of \mathcal{M} onto \mathcal{A} .

However, there exists no local, intrinsic characterization of factors M in the class \mathcal{CF} , that does not specifically use the Cartan MASA of M . Such a characterization would certainly be very interesting. It may be useful in deformation-rigidity theory, but also for studying permanence properties of \mathcal{CF} , such as stability to inductive limits, to finite index extension/restriction, or to crossed products by amenable groups. The criterion 1.4.1(2) may be of help in this direction. A related question is to find an intrinsic, local characterization of factors with unique (up to unitary conjugacy) Cartan subalgebra.

There are reasons to believe that any irreducible subfactor $N \subset M$ of a factor M in the class \mathcal{CF} has Jones index equal to the square norm of a (finite or infinite) bipartite graph. This may even be true for the (possibly larger) class of all s-thin factors. We will discuss the motivations behind this conjecture in a future paper.

5.3. Strengthened singularity. As shown in ([P81d]), any II_1 factor M with the property (T) has a MASA $A \subset M$ with the property that the only automorphisms of M that normalize A are the automorphism of M implemented by unitaries in A . With the terminology in the remark before Corollary 2.2, this amounts to A being $\text{Aut}(M)$ -singular. Equivalently, if $\theta : M \simeq N$ is an isomorphism of M onto another II_1 factor, then θ is in some sense uniquely determined by its restriction to A , $\theta|_A$. For this reasons, a MASA with this property in a II_1 factor M is called *super-singular* in M (see 5.1 in [P13]). It is shown in [P13] that, besides property (T) factors, the hyperfinite II_1 factor has super-singular MASAs as well. It is an open problem whether any separable II_1 factor has super-singular MASAs.

The proof of Corollary 2.2 shows that the following property for a MASA $A \subset N$ implies super-singularity: given any embedding of N into a II_1 factor M_0 such that $[M_0 : N] < \infty$, one has $\mathcal{N}_{M_0}(A) = \mathcal{U}(A)\mathcal{U}(N' \cap M_0)$. By ([P86]), any property (T) II_1 factor N has a MASA $A \subset N$ satisfying this strengthened super-singularity. Indeed, by [J83] any embedding with finite index $N \hookrightarrow M_0$ arises from a basic construction $M_0 = \langle N, P \rangle$, for some subfactor $P \subset N$ with $[N : P] = [M_0 : N]$, while by (Theorem 4.5.1 in [P86]) there are only countably many subfactors of finite index of N up to unitary conjugacy, so Theorem 2.1 applies.

On the other hand, Theorem 2.1 suggests the following question: does there

exist a separable II_1 factor N with a MASA $A \subset N$ having the property that for any embedding of N into a separable II_1 factor M_0 with $N' \cap M_0$ atomic, one has $\mathcal{N}_{M_0}(A) = \mathcal{U}(A)\mathcal{U}(N' \cap M_0)$? The answer to this question is however negative: if $A \subset N$ is a MASA that one also identifies with a Cartan MASA in the hyperfinite II_1 factor, $A \simeq L^\infty([0, 1]) \hookrightarrow R$, then $M_0 = N *_A R$ has the property that $N' \cap M_0 = \mathbb{C}1$ but $\mathcal{N}_{M_0}(A)'' \supset R$.

Another strengthening of the singularity property for a MASA $A \subset M$ is obtained by requiring A to be *maximal amenable* (or equivalently, maximal AFD, by [C75]) in M , i.e., to be so that there exists no intermediate amenable subalgebra $A \subset B \subset M$ with $A \neq B$ (so M must be non-amenable). The existence of such MASAs was discovered in [P81c], where it was shown that $A = L^\infty([0, 1])$ is maximal amenable in $M = A * P$. In particular, the MASA A_u generated by one of the generators of the free group $u \in \mathbb{F}_n$ is maximal amenable in $M = L(\mathbb{F}_n)$. We have conjectured in the early 1980s that any non-amenable II_1 factor contains maximal amenable MASAs. We will discuss this problem in details in a forthcoming paper.

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